§1 Definition of abelian covering and preliminary examples

"it means compact as complex analytic variety"

Def Let Y be a smooth complete algebraic variety over

"zero locus of polynom. in $\mathbb{C}^n$ or $\mathbb{P}^n$."

$\mathbb{C}$ and let G be a finite group.

A Galois cover of Y is a finite morphism $\pi: X \to Y$

"finite fibres + π is top. proper in the Euclid. top of X and Y".

, with X normal, such that G acts faithfully on X

"not so bad singularities, roughly speaking the singular locus has codimension at least 2"

and π factors as the quotient map $X \to X/G$ and an isomorphism $X/G \xrightarrow{\sim} Y$:

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d] & Y \\ X/G \arrow[ur, "\sim"'] & \end{tikzcd}$$

- We say that π is an abelian covering of Y if G is an abelian group.
- We say that π is a smooth Galois cover if X is smooth.

IMPORTANT The theory of abelian coverings works more in general on algebraically field $\mathbb{K}$ of characteristic that does NOT divide $|G|$.

However, we are going to work with $\mathbb{K} = \mathbb{C}$, and we will switch often from the Zariski Topology to the Euclidean Topology using Chow's theorem and GAGA principle.

"every analytic subspace of a complex proj. space closed in the Eucl. top is closed in the Zariski top."

We show some simple examples of Galois coverings without studying them in depth yet.

Example 0: Let us consider a complex manifold X and a finite group G acting on X.

If the action of G on X is PROPERLY DISCONTINUOUS "$\forall x \in X\ \exists U \subseteq X,\ g(U) \cap U \neq \emptyset \Rightarrow g = 1_G$".

then X/G has a structure of complex manifold and $\pi : X \to X/G$ is a holomorphic top. covering.

In our language, $\pi : X \to X/G$ is a Galois cover of $Y = X/G$ with group G.

This result is called Cartan Theorem and it may be proved in any classical Algebraic Topology course.

Remark In the language of Algebraic Geometry, we can say that in the above case, the map

$\pi : X \longrightarrow X/G$ is étale

"the ramification locus of the map is zero, namely the differential of π does NOT vanish for every point of X"

We can roughly say that Galois coverings are a natural generalization of topological coverings in Algebraic Geometry.

In particular, Galois coverings allow us to work with maps similar to topological coverings but which (possibly) have non-trivial ramification locus.
"Maps of this kind conceal a "rich geometry".

example The map $\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ x & \longmapsto & x^n \end{array}$, $n \geq 2$ is <u>NOT</u> a topological covering (because of the point $x=0$) but only $\begin{array}{ccc} \mathbb{C}\setminus\{0\} & \to & \mathbb{C}\setminus\{0\} \\ x & \longmapsto & x^n \end{array}$ is a top. covering.

Instead, $\begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ x & \longmapsto & x^n \end{array}$ is a Galois covering of $\mathbb{C}$ with group $G = \mathbb{Z}_n := \mathbb{Z}/_{n\mathbb{Z}}$.

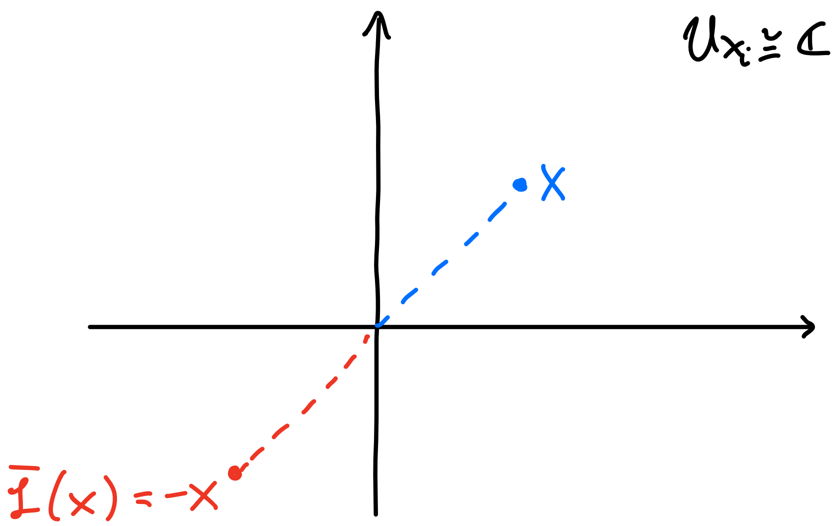
Example 1 (Double covering)

We take $X = \mathbb{P}^1(x_0, x_1)$ and $G = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

$$\bar{0} := \mathrm{Id}_X \,, \qquad \begin{aligned} I : X &\longrightarrow X \\ [x_0, x_1] &\longmapsto [x_0, -x_1] \end{aligned}$$

$V_{x_i} := \{x_i \neq 0\} \subseteq X$, then locally on U_{x_i} I is the opposite map:

$$\begin{aligned} I : \mathcal{U}_{x_i} \cong \mathbb{C} &\longrightarrow \mathcal{U}_{x_i} \cong \mathbb{C} \\ x &\longmapsto -x \,, \quad \text{where } x := \frac{x_j}{x_i} \end{aligned}$$



The action of G on $\mathbb{P}^1$ define the double quotient

$$\begin{aligned} \pi : X &\longrightarrow Y := \mathbb{P}^1(z_0, z_1) \\ [x_0, x_1] &\longmapsto [x_0^2, x_1^2] \end{aligned}$$

Exercise: Prove that π is a double quotient of Y, namely that the fibres of π corresponds to the orbits of the action of G on X.

Remark Notice that $\pi : X \to Y$ is NOT étale in this case.
Indeed $d\pi_x = \frac{d}{dx}(x^2) = 2x = 0 \iff x = 0$, so the ramification locus of π is $\mathrm{Ram}(\pi) = [1,0] + [0,1]$.

Example 2 (Bi-double cover)

two gens. of G

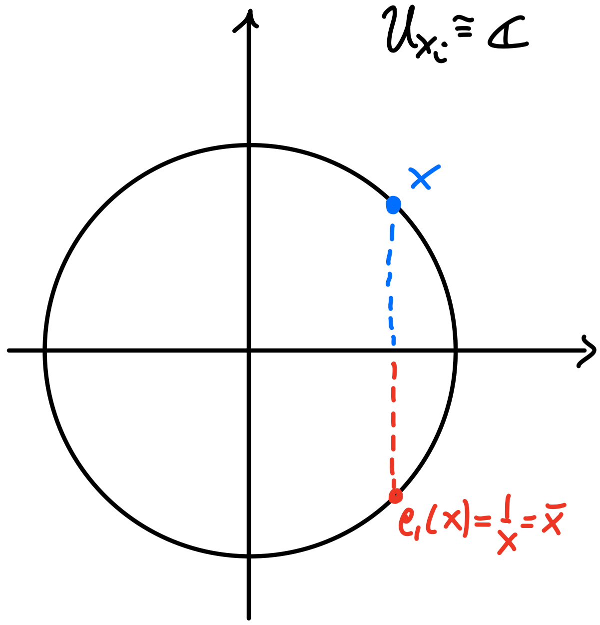
We take $X = \mathbb{P}^1(x_0, x_1)$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$

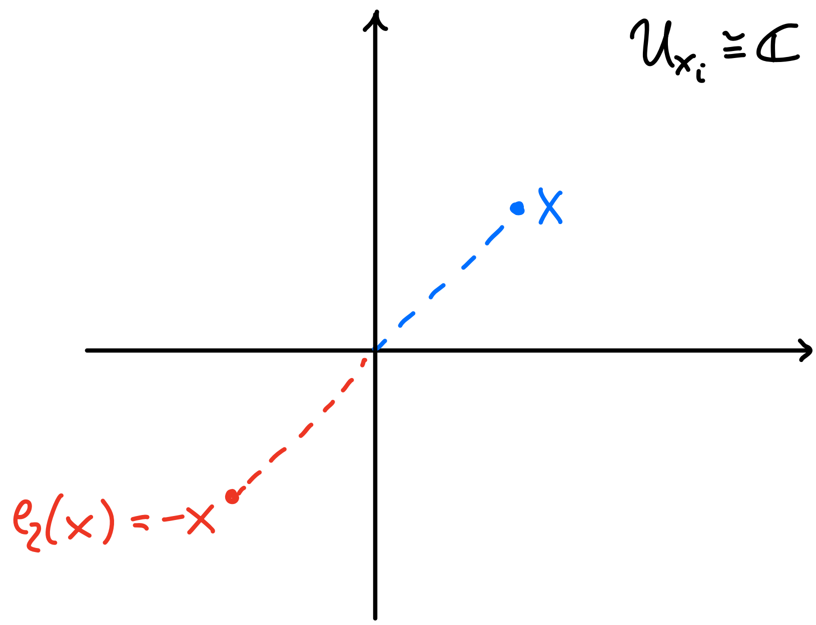
$\bar{0} := Id_X$, $\quad e_1 : X \longrightarrow X, \ [x_0, x_1] \mapsto [x_1, x_0]$, $\quad e_2 : X \longrightarrow X, \ [x_0, x_1] \mapsto [x_0, -x_1]$

On $U_{x_i} = \{x_i \neq 0\} \subseteq X$ these two maps are respectively the inverse and opposite maps:

$e_1 : x \longmapsto \frac{1}{x} \ (= \bar{x} \text{ on } S^1)$ $\quad$ where $x := \frac{x_j}{x_i}$.

$e_2 : x \longmapsto -x$





The action of G on X defines the bi-double quotient

$$\pi : X \longrightarrow Y := \mathbb{P}^1(z_0, z_1), \quad [x_0, x_1] \longmapsto [x_0^4 + x_1^4, x_0^2 x_1^2]$$

Exercise Prove that π is a double quotient of Y, namely that the fibres of π corresponds to the orbits of the action of G on X.

Example 3 (S_3-cover)

$$G = S_3 = \langle \tau, \sigma \mid \tau^2 = \sigma^3,\ \tau\sigma = \sigma^2\tau \rangle = \langle \tau, \sigma \rangle$$

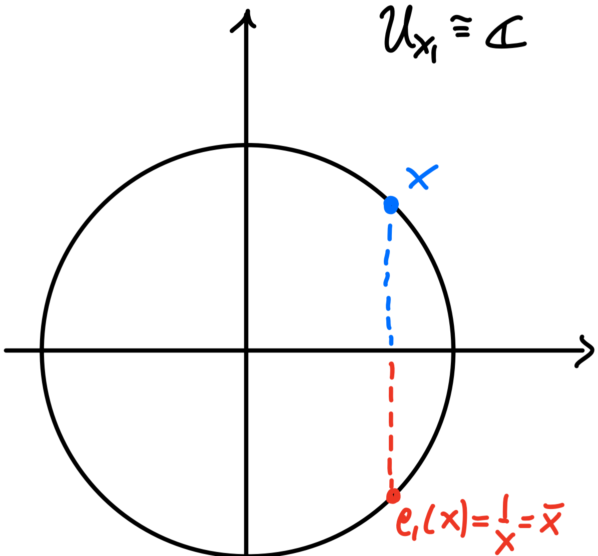
Let us consider $G = S_3 = \langle \underset{\text{transp.}}{\tau}, \underset{\text{3-cycle}}{\sigma} \rangle$ and the action on $X = \mathbb{P}^1$:

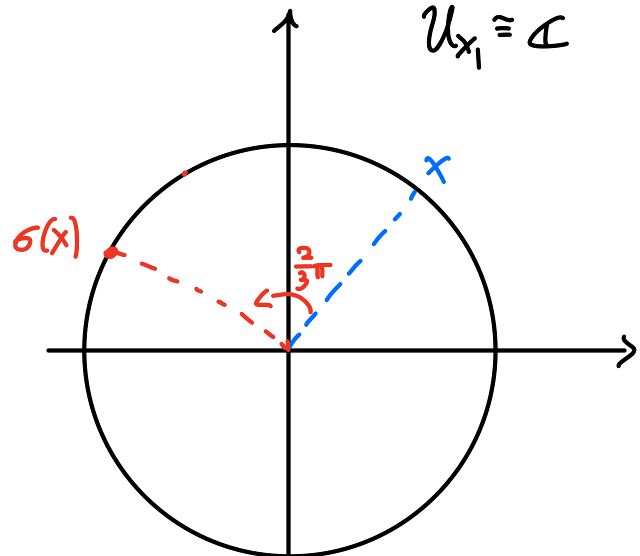
$$\tau : X \longrightarrow X, \quad [x_0, x_1] \mapsto [x_1, x_0] \qquad , \qquad \sigma : X \longrightarrow X, \quad [x_0, x_1] \mapsto [\zeta_3 x_0, x_1]$$

$\zeta_3 := e^{\frac{2\pi i}{3}}$ third root of unity

Locally around $U_{x_0} = \{x_0 \neq 0\}$ the action is

$$\tau : x \mapsto \frac{1}{x} \left(= \bar{x} \text{ on } S^1\right) \qquad , \qquad \sigma : x \mapsto \zeta_3^2 x$$





The action of S_3 on X define the S_3-quotient

$$\pi : X \longrightarrow Y := \mathbb{P}^1(z_0, z_1)$$

$$[x_0, x_1] \mapsto \left[x_0^3 x_1^3, \frac{x_0^6 + x_1^6}{2}\right]$$

Example 4 Consider $X := \mathbb{P}^2$, $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle e_1, e_2 \rangle$

and
$$e_1 : \begin{array}{c} X \longrightarrow X \\ [x_0, x_1, x_2] \mapsto [x_0, -x_1, x_2] \end{array}, \quad e_2 : \begin{array}{c} X \longrightarrow X \\ [x_0, x_1, x_2] \mapsto [x_0, x_1, -x_2] \end{array}$$

Then the quotient map is
$$\begin{array}{c} X \longrightarrow Y := \mathbb{P}^2(y_0, y_1, y_2) \\ [x_0, x_1, x_2] \mapsto [x_0^2, x_1^2, x_2^2] \end{array}$$

Example 5 (Something can go wrong)

Clearly sometimes we can quotient by a group and obtain something singular (so not a Galois covering by our definition).

Indeed, let us consider $X = \mathbb{C}^2$ with variables x, y. and $G = \mathbb{Z}_2$, that acts on X by

$$\begin{array}{c} I : X \longrightarrow X \\ (x, y) \mapsto (-x, -y) \end{array}$$

Then the invariant functions by the action of G on $\mathbb{C}[x, y]$ are x^2, y^2, xy and the homomorphism

$$\begin{array}{ccc} \mathbb{C}[X, Y, Z] & \longrightarrow & \mathbb{C}[x, x^2, y^2]) \\ X & \longmapsto & x^2 \\ Y & \longmapsto & y^2 \\ Z & \longmapsto & xy \end{array}$$

has kernel $I = (Z^2 - XY) \subseteq \mathbb{C}^3_{X,Y,Z}$, so

$$X/G = Z(z^2-xy) \subseteq \mathbb{C}^3_{x,y,z}$$

and the quotient map is

$$\begin{aligned} X &\longrightarrow X/G = Z(z^2-xy) \subseteq \mathbb{C}^3_{x,y,z} \\ (x,y) &\longmapsto (x^2, y^2, xy) \end{aligned}$$

Notice that $Z(z^2-xy)$ is singular at $p=(0,0,0)$.

(This kind of singularity is called of type A_1)

§2. Preliminaries (Griffiths–Harris, Principles of Alg. Geom.)

§2.1 Sheaves

Def Let X be a top. space, a sheaf $\mathcal{F}$ on X associates

pre-sheaf:

- to each open set $\mathcal{U} \subseteq X$ a (abelian) group $\mathcal{F}(\mathcal{U})$, called the group of sections over $\mathcal{U}$;
- to each pair $\mathcal{U} \subseteq V$ of open sets a homomorph.
$$r_{\mathcal{U}V} : \mathcal{F}(V) \to \mathcal{F}(\mathcal{U}),$$
called the restriction map, such that

1. for any $\mathcal{U} \subseteq V \subseteq W$ we have $r_{\mathcal{U},W} = r_{\mathcal{U},V} \circ r_{V,W}$; given a section $\sigma \in \mathcal{F}(V)$, we can denote $\sigma_{|\mathcal{U}} := r_{\mathcal{U},V}(\sigma)$;
2. For any pair $\mathcal{U}, V \subseteq X$ and $\sigma \in \mathcal{F}(\mathcal{U})$, $\tau \in \mathcal{F}(V)$, such that

$$\sigma_{|_{\mathcal{U}\cap V}} = \tau_{|_{\mathcal{U}\cap V}}$$

there exists a section $\rho \in \mathcal{F}(\mathcal{U}\cup V)$ with

$$\rho_{|_{\mathcal{U}}} = \sigma \quad , \quad \rho_{|_V} = \tau$$

3. If $\sigma \in \mathcal{F}(\mathcal{U}\cup V)$ and

$$\sigma_{|_{\mathcal{U}}} = \sigma_{|_V} = 0$$

then $\sigma = 0$.

<u>Examples</u> These are the usual examples of sheaves:

1. On a C^∞-manifold M we have the sheaves

$$\underset{\substack{\text{sheaf of}\\ C^\infty\text{-non}\\ \text{zero}\\ \text{functions}}}{C^\infty, \; C^*}, \quad \underset{\substack{\text{sheaf of}\\ p\text{-forms}}}{\mathcal{A}^p}, \quad \underset{\substack{\text{sheaf of}\\ \text{closed } p\text{-forms}}}{Z^p}, \quad \underbrace{\mathbb{Z}, \mathbb{Q}, \mathbb{R} \text{ and } \mathbb{C}}_{\text{sheaves of locally constant functions}}$$

2. On a complex manifold M, we also have

$$\underset{\substack{\text{sheaf of}\\ \text{holomorphic}\\ \text{functions}}}{\mathcal{O}} \qquad \underset{\substack{\text{sheaf of}\\ \text{non-zero holom.}\\ \text{functions as a}\\ \text{multiplicative group}}}{\mathcal{O}^*} \qquad \underset{\substack{\text{sheaf of}\\ \text{holomorphic}\\ p\text{-forms}}}{\Omega^p} \qquad \text{ecc...}$$

3. An important sheaf on a complex manifold M is also the sheaf of meromorphic functions $\mathcal{M}$:

$$\mathcal{M}(\mathcal{U}) = \left\{ \{(\mathcal{U}_i, \overset{\overset{f}{\shortparallel}}{g_i/h_i})\}_i : \begin{array}{l} -\{\mathcal{U}_i\}_i \text{ is a covering of } \mathcal{U} \\ -g_i, h_i \text{ are relatively prime holomorphic functions on } \mathcal{U}_i \\ -f_{|_{\mathcal{U}_i}} = f_{|_{\mathcal{U}_j}} \text{ on } \mathcal{U}_i \cap \mathcal{U}_j \text{, namely } g_i h_j = g_j h_i \text{ on } \mathcal{O}(\mathcal{U}_i \cap \mathcal{U}_j) \end{array} \right\} \Big/ \sim$$

Where $\{(U_i, {}^{g_i}/_{h_i})\}_i \sim \{(U'_i, {}^{g'_i}/_{h'_i})\} \overset{def}{\iff} \forall p \in U$ and $U_i, U'_j \ni p$ $\exists V \subseteq U_i \cap U'_j$ s.t. $g_i h'_j = g'_j h_i$ on V.

Def A map of sheaves $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ of X is a collection of homomorphisms s.t. for any $U \subseteq V$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\alpha(U)"] & \mathcal{G}(U) \\ \mathcal{F}(V) \arrow[u, "r_{U,V}"] \arrow[r, "\alpha(V)"'] & \mathcal{G}(V) \arrow[u] \end{tikzcd}$$

commutes

Def Given a map $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, the kernel sheaf of α is

$$\ker(\alpha)(U) = \ker(\alpha_U : \mathcal{F}(U) \to \mathcal{G}(U)\}$$

Warning: $\mathrm{Coker}(\alpha)(U) := \mathcal{G}(U) / \alpha(U)(\mathcal{F}(U))$ is not in general a sheaf but only a pre-sheaf (some of the previous properties 1. 2. or 3. may fail).

Def A section of $\mathrm{Coker}(\alpha)(U)$ is a pair $\{(U_i, s_i)\}_i$ with $s_i \in \mathcal{G}(U_i)$ s.t.

$$s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} \in \alpha_{U_i \cap U_j}(\mathcal{F}(U_i \cap U_j))$$

We identify two collections $\{(U_i, s_i)\}$, $(U'_i, s'_i)\}$ if $\forall p \in U$ and $U_i, U'_j \ni p$ $\exists V \subseteq U_i \cap U'_j$ s.t.

$$s_i|_V - s'_j|_V \in \alpha_V(\mathcal{F}(V))$$

Remark $\mathrm{Coker}(\alpha)$ is the fascification of the pre-sheaf $\mathcal{G}/\alpha(\mathcal{F})$.

Def A sequence
$$0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$$
is exact if $\mathcal{E} = \mathrm{Ker}(\beta)$ and $\mathcal{G} = \mathrm{Coker}(\alpha)$.

Warning: An exact sequence does <u>NOT</u> imply in general that
$$0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0 \qquad (*)$$
is exact for any open subset $U \subseteq X$.
For instance, the exponential sequence
$$0 \to \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$
is <u>NOT</u> exact in the $(*)$ sense.

[It only implies that $0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U)$ is exact and that given a section $s \in \mathcal{G}(U)$, then $\forall p \in X$ $\exists V \subseteq U$, $p \in V$, such that $s_{|V} \in \beta(\mathcal{F}(V))$.]

§2.2 Čech Cohomology

Čech cohomology is a powerful tool in Alg. Geometry; it measures the "obstructions" in a topological space.
More precisely, it measures when a collection of local data can be glued reasonably in a global datum of the space:

- The cohomology group $H^0(X, \mathcal{F})$ measures the global sections of the sheaf $\mathcal{F}$;

- $H^1(X, \mathcal{F})$ measures the obstruction to glue local data in a global datum. In particular, if
  $H^1 = 0 \Rightarrow$ any reasonable local data can be glued to a global datum

- $H^n(X, \mathcal{F})$, $n \geq 2$, measures more complicated obstructions

(a classical example is Mittag-Leffler problem)

With respect to the other kind of cohomology theories, Čech cohomology is easier computable as we can choose a suitable open cover to work with.

Furthermore, when the space X is "good" enough, then Čech cohomology is isomorphic to the usual other cohomology theories (such as singular cohomology)

This permits to use Čech cohom. for concrete calculus and then translates that computation to more abstracts coohomology theories to deduce global geometric properties.

Def Let $\mathcal{F}$ be a sheaf on X and $\underline{\mathcal{U}} = \{\mathcal{U}_\alpha\}_\alpha$ be a locally finite open cover of X.

"$\forall x \in X \; \exists V \subseteq X$ s.t. V intersects only a finite number of $\mathcal{U}_i$"

$$C^0(\underline{\mathcal{U}}, \mathcal{F}) = \prod_\alpha \mathcal{F}(\mathcal{U}_\alpha) = \{(f_i)_i : f_i \in \mathcal{F}(\mathcal{U}_i)\}$$

$$C^1(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha \neq \beta} \mathcal{F}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) = \left\{(f_{i_0 i_1})_{i_0 i_1} : f_{i_0 i_1} \in \mathcal{F}(\mathcal{U}_{i_0} \cap \mathcal{U}_{i_1})\right\}$$

$$\vdots$$

$$C^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{i_0 \neq i_1 \neq \dots \neq i_p} \mathcal{F}(\mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_p})$$

an element $\sigma = \{\sigma_I \in \mathcal{F}(\bigcap_{i \in I} \mathcal{U}_i)\}_{\#I = p+1}$ ($I = i_0 i_1 \dots i_p$) is called a p-cochain of $\mathcal{F}$.

The coboundary operator is

$$\delta : C^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$$

where
$$(\delta\sigma)_{i_0 i_1 \dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \left(\sigma_{i_0 \dots \hat{i_j} \dots i_{p+1}}\right)\Big|_{\mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_{p+1}}}$$

Example: $\sigma \in C^0(\underline{\mathcal{U}}, \mathcal{F}) \Rightarrow$

$$(\delta\sigma)_{\alpha\beta} = -\sigma_\alpha + \sigma_\beta \quad \text{on } U_\alpha \cap U_\beta$$

$\sigma \in C^1(\underline{\mathcal{U}}, \mathcal{F}) \Rightarrow$

$$(\delta\sigma)_{\alpha\beta\gamma} = \sigma_{\beta\gamma} - \sigma_{\alpha\gamma} + \sigma_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma$$

A p-cocycle is $\sigma \in C^p(\underline{\mathcal{U}}, \mathcal{F})$ s.t. $\delta\sigma = 0$.

Exercise Prove that a p-cocycle has to satisfy the skew-symmetric condition

$$\sigma_{i_0 \dots i_p} = -\sigma_{i_0 \dots i_{q-1} i_{q+1} i_q i_{q+2} \dots i_p}$$

so, for instance, we have $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$ on $C^2(\underline{\mathcal{U}}, \mathcal{F})$ for a p-cocycle.

A p-cochain σ is called coboundary if

$$\sigma = \delta\tau \quad \text{for some } \tau \in C^{p-1}(\underline{\mathcal{U}}, \mathcal{F}).$$

Prop: $\delta^2 = 0$, namely, denoted by

$$Z^p(\underline{\mathcal{U}}, \mathcal{F}) := \ker(\delta : C^p \to C^{p+1})$$

then $\delta C^{p-1}(\underline{\mathcal{U}}, \mathcal{F}) \subseteq Z^p(\underline{\mathcal{U}}, \mathcal{F})$.

Finally, we define $H^p(\underline{\mathcal{U}}, \mathcal{F}) := \dfrac{Z^p(\underline{\mathcal{U}}, \mathcal{F})}{\delta C^{p-1}(\underline{\mathcal{U}}, \mathcal{F})}$

Remark Clearly, $H^p(\underline{\mathcal{U}}, \mathcal{F})$ depends by the choice of $\underline{\mathcal{U}}$.

One can define Čech cohomology in an abstract way using direct limit.

However, we prefer to give a more practical definit.

Def Let $\underline{\mathcal{U}}$ be an acyclic covering of X for the sheaf $\mathcal{F}$. "a good cover; all the covering that we will consider are acyclic"

The p-th Čech cohomology group is

$$H^p(X, \mathcal{F}) := H^p(\underline{\mathcal{U}}, \mathcal{F})$$

Remark Given two acyclic coverings $\underline{\mathcal{U}}$ and $\underline{\mathcal{U}}'$, then

$$H(\underline{\mathcal{U}}, \mathcal{F}) \cong H(\underline{\mathcal{U}}', \mathcal{F})$$

so that the definition above is well-posed and it does not depend by the choice of the acyclic covering $\underline{\mathcal{U}}$.

Examples 1) Let $X = \mathbb{P}^1$, $\mathcal{F} = \mathcal{O}$; we choose

$$\mathcal{U}_{x_0} = \{x_0 \neq 0\}, \quad \mathcal{U}_{x_1} = \{x_1 \neq 0\}$$

One can prove $\underline{\mathcal{U}} = \{\mathcal{U}_{x_0}, \mathcal{U}_{x_1}\}$ is an acyclic covering for the sheaf $\mathcal{O}$, so

$$H^p(\mathbb{P}^1, \mathcal{O}) = H^p(\underline{\mathcal{U}}, \mathcal{O})$$

We define by $u := \frac{X_1}{X_0}$ on U_{X_0} and $v := \frac{X_0}{X_1}$ on U_{X_1}.

$$C^0(\underline{U}, \mathcal{O}) = \{(f,g) : f \in \mathcal{O}(U_{X_0}), g \in \mathcal{O}(U_{X_1})\}$$

In general, we can write

$$f = \sum_{n=0}^{\infty} a_n u^n \text{ on } U_{X_0}, \quad g = \sum_{n=0}^{\infty} b_n v^n \text{ on } U_{X_1}$$

Thus a cocycle (f,g) needs to satisfy

$$\delta(f,g) = g - f = 0 \quad \text{on } U_{X_0} \cap U_{X_1}$$

$$\sum_{n=-\infty}^{+\infty} \gamma_n u^n \quad \text{where } \gamma_n = \begin{cases} -a_n & \text{if } n>0 \\ a_0 - b_0 & \text{if } n=0 \\ b_{-n} & \text{if } n<0 \end{cases}$$

which is zero $\Leftrightarrow$ $\begin{matrix} a_n = b_n = 0 & \forall n>0 \\ a_0 = b_0 \end{matrix}$ $\quad f = g = a_0$

$\Rightarrow H^0(\mathbb{P}^1, \mathcal{O}) \cong \mathbb{C}$ (as we expected)

Let us consider

$$C^1(\underline{U}, \mathcal{O}) = \{h \in \mathcal{F}(U_{X_0} \cap U_{X_1})\} = \mathcal{F}(U_{X_0} \cap U_{X_1})$$

Clearly $C^1(\underline{U}, \mathcal{O}) = Z^1(\underline{U}, \mathcal{O})$ as we only have 2 open subsets of $\mathbb{P}^1$

Instead, let us consider an element

$$h \in \mathcal{F}(U_{X_0} \cap U_{X_1}), \quad h = \sum_{n=-\infty}^{+\infty} \gamma_n u^n = \sum_{n=-\infty}^{+\infty} \gamma_n v^{-n}$$

Then we can define $f = -\sum_{n=0}^{\infty} \gamma_n u^n$ on U_{X_0} and $g = \sum_{n=1}^{\infty} \gamma_{-n} v^n$ on U_{X_1}

and obtain $h = \delta(f,g)$

$$\Rightarrow Z^1(\underline{\mathcal{U}}, \mathcal{O}) = \delta C^0(\underline{\mathcal{U}}, \mathcal{O}) \Rightarrow H^1(\mathbb{P}^1, \mathcal{O}) = 0.$$

Thus $$H^p(\mathbb{P}^1, \mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } p=0 \\ 0 & \text{if } p \geq 1 \end{cases}.$$

2) Let $X = \mathbb{P}^1$, $\mathcal{F} = \Omega^1$; we still have $\underline{\mathcal{U}} = \{\mathcal{U}_{x_0}, \mathcal{U}_{x_1}\}$ is acyclic for Ω^1, so

$$H^p(\mathbb{P}^1, \Omega^1) = H^p(\underline{\mathcal{U}}, \Omega^1) \quad \forall p \geq 0$$

In this case

$$C^0(\underline{\mathcal{U}}, \Omega^1) = \{(f\,du, g\,dv) : f \in \mathcal{O}(\mathcal{U}_{x_0}), g \in \mathcal{O}(\mathcal{U}_{x_1})\}$$

Then $\delta(f\,du, g\,dv) = g\,dv - f\,du = (-gu^{-2} - f)\,du$

$v = \frac{1}{u}$ so $dv = -u^{-2}du$ on $\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1}$

so that if $f = \sum_{n=0}^{\infty} a_n u^n$, $g = \sum_{n=0}^{\infty} b_n v^n$, then

$$-gu^{-2} - f = \sum_{n=-\infty}^{+\infty} \gamma_n u^n \quad \text{where } \gamma_n = \begin{cases} -a_n & \text{if } n \geq 0 \\ 0 & \text{if } n = -1 \\ -b_{-2-n} & \text{if } n \leq -2 \end{cases}$$

Thus, $\delta(f\,du, g\,dv) = 0 \Leftrightarrow a_n = b_n = 0 \Rightarrow$

$$Z^0(\underline{\mathcal{U}}, \Omega^1) = 0 \Rightarrow H^0(\mathbb{P}^1, \Omega^1) = 0.$$

Instead, consider

$$C^1(\underline{\mathcal{U}}, \Omega^1) = \{ h\,du : h \in \mathcal{U}_{x_0} \cap \mathcal{U}_{x_1} \} = \Omega^1(\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1})$$

Clearly $Z^1(\underline{\mathcal{U}}, \Omega^1) = C^1(\underline{\mathcal{U}}, \Omega^1)$; let us find $\delta C^0(\underline{\mathcal{U}}, \Omega^1)$.

Given $\delta(f\,du, g\,dv) = (g u^{-2} - f)\,du$, we have seen above

$$\gamma_n = \begin{cases} -a_n & \text{if } n \geq 0 \\ 0 & \text{if } n = -1 \\ -b_{-2-n} & \text{if } n \leq -2 \end{cases}$$

so an element in $\delta C^0(\underline{\mathcal{U}}, \Omega^1)$ is characterized to have $\gamma_{-1} = 0$. We have obtained

$$\delta C^0(\underline{\mathcal{U}}, \Omega^1) = \{ \underbrace{h}_{\sum_{n=-\infty}^{+\infty} \gamma_n u^n} du : \gamma_{-1} = 0 \}.$$

This means that if we consider the linear map

$$\begin{aligned} Z^1(\underline{\mathcal{U}}, \Omega^1) &\longrightarrow \mathbb{C} \\ h\,du &\longmapsto \gamma_{-1} \end{aligned},$$

(the map is surjective as $\frac{1}{u}du$ on $\mathcal{U}_{x_0} \cap \mathcal{U}_{x_1}$ maps to $\gamma_{-1} = 1$)

then the kernel is $\delta C^0(\underline{\mathcal{U}}, \Omega^1)$, and so

$$H^1(\mathbb{P}^1, \Omega^1) \cong \frac{Z^1(\underline{\mathcal{U}}, \Omega^1)}{\delta C^0(\underline{\mathcal{U}}, \Omega^1)} \cong \mathbb{C}$$

We have obtained

$$H^p(\mathbb{P}^1, \Omega^1) = \begin{cases} 0 & \text{if } p = 0, p \geq 2 \\ \mathbb{C} & \text{if } p = 1 \end{cases}$$

Exercise Prove that

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & \text{if } p = q \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Remark 2 When $\mathcal{F}$ is a sheaf of vector spaces then $H^p(\underline{\mathcal{U}}, \mathcal{F})$ is a vector space. There is no reason why it should be a finite dimensional vector space!

For instance, $H^1(\mathbb{C}, \mathcal{O})$ is a infinite dimens. vec. space.

Thm 1 (Grothendieck's Vanishing Thm) (it holds also in a more gen. setting)

Let X be a projective variety and $\mathcal{F}$ be a sheaf of ab. groups. Then

$$H^p(X, \mathcal{F}) = 0 \quad \forall p > \dim(X)$$

Thm 2 (Cartan – Serre Finiteness Thm)

Let X be a projective variety and $\mathcal{F}$ be a coherent sheaf. Then all $H^p(X, \mathcal{F})$ are finite dimensional $\mathbb{C}$-vector spaces.

"a good sheaf, all of our sheaves will be coherent"

Def Given a projective variety X and a coherent sheaf $\mathcal{F}$, then [$h^i(X,\mathcal{F}) := \dim H^i(X,\mathcal{F})$]

$$\chi(\mathcal{F}) := \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, \mathcal{F})$$

is called the Euler Characteristic of the sheaf $\mathcal{F}$.

Rem. Clearly we can use the same def just when $H^i(X,\mathcal{F})$ are f.g. ab. groups and the above summand is finite. In that case $h^i(X,\mathcal{F}) := rk(H^i(X,\mathcal{F}))$.

Consequence of Universal Coefficient Thm.

Example: $\mathcal{F} = \mathbb{Z}$ is equal to the Top. Euler characteristic $e(X)$

Def $\mathcal{F} = \mathcal{O}$ is called the Euler characteristic of the structure sheaf $\mathcal{O}$.

Def The geometric genus of a projective variety X is

$$p_g(X) := h^0(X, \Omega^n_X)$$

where $n = \dim(X)$

Def The irregularity $q(X)$ of X is

$$q(X) := h^1(X, \mathcal{O}_X) \left(\underset{\text{Hodge Theory}}{=} h^0(X, \Omega^1_X) \right)$$

Thus we actually attach to any projective variety X three invariants (that are actually birational invariants)

$$\chi(\mathcal{O}_X), \quad q(X), \quad p_g(X)$$

+ the topological invariant $e(X)$.

IMPORTANT PROPERTY

Let $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0$ be an exact sequence. Then we have natural maps

$$\begin{array}{ccc} C^p(\underline{\mathcal{U}}, \mathcal{E}) & \xrightarrow{\alpha} & C^p(\underline{\mathcal{U}}, \mathcal{F}) \\ \delta \downarrow & & \downarrow \delta \\ C^{p+1}(\underline{\mathcal{U}}, \mathcal{E}) & \xrightarrow{\alpha} & C^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) \end{array} \quad , \quad \begin{array}{ccc} C^p(\underline{\mathcal{U}}, \mathcal{F}) & \xrightarrow{\beta} & C^p(\underline{\mathcal{U}}, \mathcal{G}) \\ \delta \downarrow & & \downarrow \delta \\ C^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) & \xrightarrow{\beta} & C^{p+1}(\underline{\mathcal{U}}, \mathcal{G}) \end{array}$$

So this induces maps in cohomology

$$H^p(\alpha) : H^p(X, \mathcal{E}) \longrightarrow H^p(X, \mathcal{F})$$
$$H^p(\beta) : H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G})$$

It also naturally arises the so called coboundary map:

$$H^p(X, \mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(X, \mathcal{E})$$

Theorem The sequence

$$\begin{aligned} 0 &\to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \xrightarrow{\delta^*} \\ &\to H^1(X, \mathcal{E}) \longrightarrow H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \xrightarrow{\delta^*} \\ &\to \cdots \\ &\xrightarrow{\delta^*} H^p(X, \mathcal{E}) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{G}) \to \cdots \end{aligned}$$

is exact.